Ch-6

# Linear Transformations

**Definition** Linear Transformation

Let $V$ and $W$ be two vector space over $\mathbb{R}$

A function $T: V \to W$

is called linear transformation from $V$ to $W$

if it satisfies the following properties:

① $T(v_1 + v_2) = T(v_1) + T(v_2)$ for all $v_1, v_2 \in V$

② $T(\alpha v) = \alpha T(v)$ for all $\alpha \in \mathbb{R}$, all $v \in V$

Equivalently, say,

$T(\alpha v_1 + \beta v_2) = \alpha T(v_1) + \beta T(v_2)$ for all $v_1, v_2 \in V$, all $\alpha, \beta \in \mathbb{R}$

Example ① zero Linear transformation

Let $V$ & $W$ vector space

mapping $T: V \to W$

defined by $T(v) = 0_w$ for all $v \in V$ — ①

where $0_w$ is zero vector of $W$.

Show: $T$ is L.T

(i) $T(v_1 + v_2) = T(v_1) + T(v_2)$ for all $v_1, v_2 \in V$

consider $T(v_1 + v_2) = 0_w$ [by ①]

$= 0_w + 0_w$

$= T(v_1) + T(v_2)$ [by ①]

(ii) $T(\alpha v) = \alpha T(v)$ for all $\alpha \in \mathbb{R}$, all $v \in V$

consider $T(\alpha v) = 0_w$ [by ①]

$= \alpha \cdot 0_w$

$= \alpha T(v)$

$\therefore$ zero function is Linear Transformation.

Example ② Let $V = M_{m \times n}$ (space of $m \times n$ matrices)
$W = M_{mn}$ (space of all $n \times m$ matrices)

Consider $T: V \to W$ defined by
$$T(A) = A^T \text{ for all } A \in V \quad \text{— ①}$$
to show: $T$ is linear transformation

(Solution) let $A_1$ & $A_2$ be any two matrices $V = M_{m \times n}$

then $T(A_1 + A_2) = (A_1 + A_2)^T$ [by ①]
$= A_1^T + A_2^T$
$= T(A_1) + T(A_2)$

also $T(\alpha A) = (\alpha A)^T$ — [by ①]
$= \alpha A^T$
$= \alpha T(A)$ for any $\alpha \in \mathbb{R}$, $A \in V$

$\therefore$ $T$ is L.T (linear transformation)

Definition Linear Operator

let $V$ be vector space. Consider the mapping $T: V \to V$ is called Linear Operator.

Remark: A linear operator is a L.T (linear transformation) from vector space to itself.

Example ⑤ Identity Linear Operator

Let $V$ be V.S (vector space)
Consider $T: V \to V$ defined by $T(v) = v$ — ① for all $v \in V$

(Show:) $T$ is linear operator

(i) (show) $T(v_1 + v_2) = T(v_1) + T(v_2)$ for all $v_1, v_2 \in V$

consider $T(v_1 + v_2) = v_1 + v_2$ — [by ①]
$= T(v_1) + T(v_2)$

and $T(\alpha v_1) = \alpha v_1$ [by ①]
$= \alpha T(v_1)$

∴ T is Linear operator
Also known as Identity linear operator.

Example (6) Contractions & dilations
Show: let $k \in \mathbb{R}$, Define $T: \mathbb{R}^n \to \mathbb{R}^n$
s.t $T(v) = kv$ for all $v \in \mathbb{R}^n$
is Linear operator
(Solution) (Do as excercise)

Remark: T is Linear operator called dilation or contraction according as $|k| > 1$ or $|k| < 1$ respectively.

Example (7) Projections
Consider the mapping $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by
$T([x_1, x_2, x_3]) = [x_1, x_2, 0]$ — (1), where $[x_1, x_2, x_3] \in \mathbb{R}^3$
Show: T is Linear operator.

(Solution): let $v_1 = [x_1, x_2, x_3]$
$v_2 = [y_1, y_2, y_3] \in \mathbb{R}^3$

then $T[v_1 + v_2] = T([x_1, x_2, x_3] + [y_1, y_2, y_3])$
$= T([x_1 + y_1, x_2 + y_2, x_3 + y_3])$
$= [x_1 + y_1, x_2 + y_2, 0]$ [by (1)]
$= [x_1, x_2, 0] + [y_1, y_2, 0]$
$= T[x_1, x_2, x_3] + T[y_1, y_2, y_3]$
$= T(v_1) + T(v_2)$ for all $v_1, v_2 \in \mathbb{R}^3$

Consider $T(\alpha v_1)$ where $\alpha \in \mathbb{R}$, $v_1 \in \mathbb{R}^3$
$T(\alpha v_1) = T(\alpha [x_1, x_2, x_3])$
$= T([\alpha x_1, \alpha x_2, \alpha x_3])$
$= [\alpha x_1, \alpha x_2, 0]$ [by (1)]

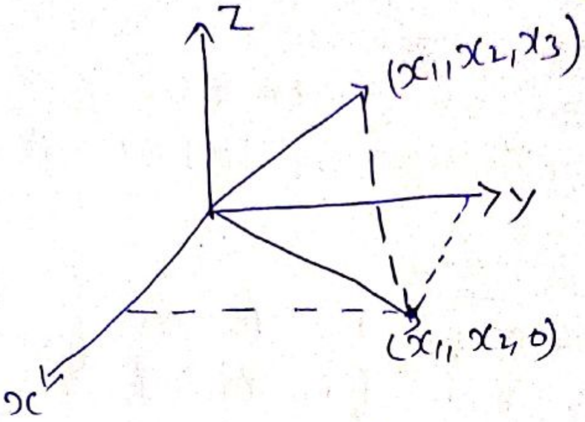
$= \alpha\,[x_1, x_2, 0]$

$= \alpha T([x_1, x_2, x_3])$ [by ①]

$= \alpha\,T(v)$

The T (above) linear operator is called as projection operator.

Since the above T operator, projects each vector in $\mathbb{R}^3$ to a corresponding vector in $xy$-plane




Example ⑪ Shear operators.

Let $k$ be a fixed scalar in $\mathbb{R}$,

Consider the function $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} x + ky \\ y \end{bmatrix} \quad — ①$$

Show T is linear operator

(Solution) Let $v_1 = [x_1, y_1]$ and $v_2 = [x_2, y_2]$ be two vectors in $\mathbb{R}^2$

then $T(v_1 + v_2) = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}(v_1 + v_2)$

$= \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} v_1 + \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} v_2$ [by ①]

$= T(v_1) + T(v_2)$

Let $\alpha \in \mathbb{R}$ and $v = [x, y] \in \mathbb{R}^2$

then

$$T(\alpha v) = \begin{bmatrix} 1 & K \\ 0 & 1 \end{bmatrix} (\alpha v)$$

$$= \alpha \left( \begin{bmatrix} 1 & K \\ 0 & 1 \end{bmatrix} v \right)$$

$$= \alpha T(v)$$

Hence, T is linear operator, called a shear in x-axis with factor K.

Theorem (6.1) <u>Properties of Linear Transformation</u>

Let V and W be two vector space and let $T : V \to W$ be a linear transformation.

Let $O_v$ be zero vector in V and $O_w$ be zero vector in W.

then ① $T(O_v) = O_w$

② $T(-v) = -T(v)$ for all $v \in V$

③ $T(u - v) = T(u) - T(v)$ for all $u, v \in V$

④ $T(a_1 v_1 + a_2 v_2 + \cdots + a_n v_n)$
$= a_1 T(v_1) + a_2 T(v_2) + \cdots + a_n T(v_n)$
for all $a_1, a_2, \ldots a_n \in \mathbb{R}$
$v_1, v_2, \ldots v_n \in V$
for $n \geq 1$.

proof) ① $T(O_v) = T(0 \cdot O_v)$
$= 0 (T(O_v))$ (by property of Linear transformation)
$= O_w$

② $T(-v) = T((-1) v)$
$= (-1) T(v)$ [by property $T(\alpha v) = \alpha T(v)$]
$= -T(v)$

③ $T(u-v) = T(u+(-1)v)$

$= T(u) + T((-1)v)$ by property of L.T $[T(v_1+v_2) = T(v_1) + T(v_2)]$

$= T(u) - T(v)$

④ We prove by induction on $n$.

For $n=1$, we have

$T(a_1 v_1) = a_1 T(v_1)$ [by property of L.T $T(\alpha v_1) = \alpha T(v_1)$]

So, result is true for $n=1$

Similarly for $n=2$, we have

$T(a_1 v_1 + a_2 v_2) = T(a_1 v_1) + T(a_2 v_2)$ by property $[T(v_1+v_2) = T(v_1)+T(v_2)]$

$= a_1 T(v_1) + a_2 T(v_2)$ [by property $T(\alpha v_1) = \alpha T(v_1)$]

true for $n=2$

let result holds for $n=m$

to show: for $n=m+1$

Consider

$T(a_1 v_1 + a_2 v_2 + \cdots + a_m v_m + a_{m+1} v_{m+1})$

$= T(a_1 v_1 + a_2 v_2 + \cdots + a_m v_m) + T(a_{m+1} v_{m+1})$ (result holds for $n=m$)

$= a_1 T(v_1) + a_2 T(v_2) + \cdots + a_{m+1} T(v_{m+1})$ (by induction hypothesis)

∴ result is true for $n=m+1$

## Composition of Linear Transformation

If $f: X \to Y$ and $g: Y \to Z$ are functions then the composition of $f$ and $g$ defined to be function $g \circ f: X \to Z$

given by $(g \circ f)(x) = g(f(x))$

Theorem 6.2 let $V$, $W$, $X$ be vector space.
Let $T_1 : V \to W$ and $T_2 : W \to X$ be linear transformations.
Then the composition function $T_2 \circ T_1 : V \to X$
given by $(T_2 \circ T_1)(v) = T_2(T_1(v))$ for all $v \in V$,
is linear transformation.

Proof) Show: $T_2 \circ T_1$ is linear transformation.

consider, $(T_2 \circ T_1)(v_1 + v_2) = T_2[T(v_1 + v_2)]$

$= T_2[T(v_1) + T(v_2)]$ (by definition of composition)

$= T_2(T(v_1)) + T_2(T_1(v_2))$ ($\because T_2$ is L.T linear transformation)

$= (T_2 \circ T_1)(v_1) + (T_2 \circ T_1)(v_1)$ (By definition of composition)

consider,
$(T_2 \circ T_1)(\alpha v) = T_2(T_1 \alpha(v))$

$= T_2(\alpha T_1(v))$ (definition of composition)

$= \alpha(T_2(T_1(v)))$ ($\because T_1$ is L.T)

$= \alpha(T_2 \circ T_1)(v)$ (definition of composition)

$\therefore T_2 \circ T_1$ is L.T

Linear Transformation and subspaces

let $V_1$ & $V_2$ be vector spaces and
$T : V_1 \to V_2$ be linear transformation
Given a set $U \subseteq V_1$
the image of $U$ in $V_2$ is defined set
$T(U) = \{T(u) : u \in U\}$

Similarly, given set $W \subseteq V_2$
the pre-image of $W$ in $V_1$ is defined
set $T^{-1}(W) = \{v \in V_1, T(v) \in W\}$

Theorem 6.3
let $V_1$ and $V_2$ be vector space and let
$T: V_1 \rightarrow V_2$ be linear transformation.
① If $U$ is a subspace of $V_1$
then $T(U)$, the image of $U$ in $V_2$, is subspace
of $V_2$.
② If $W$ is subspace of $V_2$ then $T^{-1}(W)$, the
pre image of $W$ in $V_1$, is subspace of $V_1$..

proof ① Since $U$ is subspace of $V_1$
$0_{V_1} \in U$ ( $0_{V_1}$ is zero vector of $V_1$)

By part ① of theorem 6.1, we have
$0_{V_2} = T(0_{V_1}) \in T(U)$

$\therefore T(U)$ is non-empty.
(to show:) $T(U)$ is subspace of $V_2$
let $w_1, w_2$ be any two vectors in $T(U)$
by definition of $T(U)$, we have
$w_1 = T(u_1)$ & $w_2 = T(u_2)$ for some $u_1, u_2 \in U$

So, $w_1 + w_2 = T(u_1) + T(u_2)$
$= T(u_1 + u_2)$ ($\because T$ is L.T)

Since, $U$ is subspace of $V$
$\Rightarrow u_1 + u_2 \in U$
$\therefore w_1 + w_2 \in T(U)$

let $w$ be any vector in $T(U)$ 

let $\alpha \in \mathbb{R}$

to show: $\alpha w \in T(U)$

By definition $w \in T(u)$ for some $u \in U$

then $\alpha w = \alpha T(u)$

$= \alpha T(u)$

$= T(\alpha u)$ ($\because$ T is L.T)

$\therefore T(U)$ is subspace of $V_2$

(2) Since $T(0_{v_1}) = 0_{v_2} \in W$

So, $0_{v_1} \in T^{-1}(W)$

Also, let $v_1, v_2 \in T^{-1}(W)$

$\Rightarrow T(v_1), T(v_2) \in W$

$\Rightarrow T(v_1) + T(v_2) \in W$

$\Rightarrow T(v_1 + v_2) \in W$

$\Rightarrow v_1 + v_2 \in T^{-1}(W)$

Also, let $v \in T^{-1}(W)$ ; let $\alpha \in \mathbb{R}$

then $v \in T^{-1}(W)$

$\Rightarrow T(v) \in W$

$\Rightarrow \alpha T(v) \in W$

$\Rightarrow T(\alpha v) \in W$

$\Rightarrow \alpha v \in T^{-1}(W)$

Hence $T^{-1}(W)$ is subspace of $V_1$.